Classification of Spherically Symmetric Static Spacetimes according to their Matter Collineations

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Abstract

The spherically symmetric static spacetimes are classified according to their matter collineations. These are studied when the energy-momentum tensor is degenerate and also when it is non-degenerate. We have found a case where the energy-momentum tensor is degenerate but the group of matter collineations is finite. For the non-degenerate case, we obtain either four, five, six or ten independent matter collineations in which four are isometries and the rest are proper. We conclude that the matter collineations coincide with the Ricci collineations but the constraint equations are different which on solving can provide physically interesting cosmological solutions.

Keywords : Matter symmetries, Spherically symmetric spacetimes

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1 Introduction

Einstein's field equations (EFEs) are given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3), \tag{1}$$

where G_{ab} are the components of the Einstein tensor, R_{ab} those of the Ricci and T_{ab} of the matter (energy-momentum) tensor. Also, $R = g^{ab}R_{ab}$ is the Ricci scalar, κ is the gravitational constant and, for simplicity, we take $\Lambda = 0$.

Let (M, g) be a spacetime, i.e., M is a four-dimensional, Hausdorff, smooth manifold, and g is smooth Lorentz metric of signature (+ - - -) defined on M. The manifold M and the metric g are assumed smooth (C^∞). We shall use the usual component notation in local charts, and a covariant derivative with respect to the symmetric connection Γ associated with the metric g will be denoted by a semicolon and a partial derivative by a comma.

In general relativity (GR) theory, the Einstein tensor G_{ab} plays a significant role, since it relates the geometry of spacetime to its source. The GR theory, however, does not prescribe the various forms of matter, and takes over the energy-momentum tensor T_{ab} from other branches of physics.

Collineations are geometrical symmetries which are defined by a relation of the form

$$\pounds_\xi \Phi = \Lambda, \tag{2}$$

where $\pounds$ is the Lie derivative operator, ξ^a is the symmetry or collineation vector, Φ is any of the quantities $g_{ab}, \Gamma^a_{bc}, R_{ab}, R^a_{bcd}$ and geometric objects constructed by them and Λ is a tensor with the same index symmetries as Φ. One can find all the well-known collineations by requiring the particular forms of the quantities Φ and Λ. For example if we take $\Phi_{ab} = g_{ab}$ and $\Lambda_{ab} = 2\psi g_{ab}$, where $\psi(x^a)$ is a scalar function, this defines a Conformal Killing vector (CKV) and it specializes to a Special Conformal Killing vector (SCKAV) when $\psi_{;ab} = 0$, to a Homothetic vector field when $\psi = \textit{constant}$ and to a Killing vector (KV) when $\psi = 0$. If we take $\Phi_{ab} = R_{ab}$ and $\Lambda_{ab} = 2\psi R_{ab}$ the symmetry vector ξ^a is called a Ricci inheritance collineation (RIC) and reduces to a Ricci collineation (RC) for $\Lambda_{ab} = 0$. When $\Phi_{ab} = T_{ab}$ and $\Lambda_{ab} = 2\psi T_{ab}$, where T_{ab} is the energy-momentum tensor, the vector ξ^a is called a Matter inheritance collineation (MIC) and it reduces to a Matter collineation (MC) for $\Lambda_{ab} = 0$. In the case of CKVs, the function ψ is called the conformal factor and in the case of inheriting collineations the inheriting factor.

Collineations can be proper (non-trivial) or improper (trivial). In this paper, we will define a proper MC to be an MC which is not a KV, or a HV. The MC equation can be written as

$$\pounds_\xi T_{ab} = 0 \quad \Leftrightarrow \quad \pounds_\xi G_{ab} = 0, \tag{3}$$

or in component form

$$T_{ab,c}\xi^c + T_{ac}\xi^c_{,b} + T_{cb}\xi^c_{,a} = 0. \tag{4}$$

Collineations other than motions (KVs) can be considered as non-Noetherian symmetries and can be associated with constants of motion and, up to the level of CKVs, they can be used to simplify the metric [1]. For example, Affine vectors (AVs) are related to conserved quantities [2], RCs, are related to the conservation of particle number in Friedmann Robertson-Walker spacetimes [3]. and the existence of Curvature collineations (CCs) implies conservation laws for null electromagnetic fields [4]. The set of collineations of a spacetime can be related with an inclusion relation leading to a tree like inclusion diagram [4] which shows their relative hierarchy. A collineation of a given type is proper if it does not belong to any of the subtypes in this diagram. In order to relate a collineation to a particular conservation law and its associated constant(s) of motion, the properness of the collineation must first be assured.

The motivation for studying MCs can be discussed as follows. When we find exact solutions to the Einstein's field equations, one of the simplifications we use is the assumption of certain symmetries of the spacetime metric. These symmetry assumptions are expressed in terms of isometries expressed by the spacetimes, also called Killing vectors which give rise to conservation laws [1,5]. Symmetries of the energy-momentum tensor provide conservation laws on matter fields. These symmetries are called matter collineations. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [6]. In other words, given the metric tensor of a spacetime, one can find symmetry for the physical fields describing the material content of that spacetime. There is also a purely mathematical interest of studying the symmetry properties of a given geometrical object, namely the Einstein tensor, which arises quite naturally in the theory of GR. Since it is related, via the Einstein field equations, to the material content of the spacetime, it has an important role in this theory.

Recently, there is a growing interest in the study of MCs [7-13]. Carot, et al [8] has discussed MCs from the point of view of the Lie algebra of vector fields generating them and, in particular, he discussed spacetimes with a degenerate T_{ab}. Hall, et al [9], in the discussion of RC and MC, have argued that the symmetries of the energy-momentum tensor may also provide some extra understanding of the the subject which has not been provided by Killing vectors, Ricci and Curvature collineations.

In this paper, we study the problem of calculating MCs for static spherically symmetric spacetimes for both degenerate and non-degenerate energy-momentum tensors and establish the relation between KVs, RCs and MCs. The breakdown of the paper follows. In the next section we write down MC equations for static spherically symmetric spacetimes. In section three, we shall solve these MC equations when the energy-momentum tensor is degenerate and in the next section MC equations are solved for the non-degenerate energy-momentum tensor. Finally, a summary of the results obtained will be presented.

2 Matter Collineation Equations

In this section, we write down the MC equations for spherically symmetric static spacetimes. The most general spherically symmetric metric is given as

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{\mu(t,r)}d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\theta^2$. Since we are dealing with static spherically symmetric spacetimes, Eq.(5) reduces to

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - e^{\mu(r)}d\Omega^2, \tag{6}$$

We can write MC Eqs.(4) in the expanded form as follows

$$T_{0,1}\xi^1 + 2T_0\xi^0_{,0} = 0, \tag{7}$$

$$T_0\xi^0_{,1} + T_1\xi^1_{,0} = 0, \tag{8}$$

$$T_0\xi^0_{,2} + T_2\xi^2_{,0} = 0, \tag{9}$$

$$T_0\xi^0_{,3} + \sin^2\theta T_2\xi^3_{,0} = 0, \tag{10}$$

$$T_{1,1}\xi^1 + 2T_1\xi^1_{,1} = 0, \tag{11}$$

$$T_1\xi^1_{,2} + T_2\xi^2_{,1} = 0, \tag{12}$$

$$T_1\xi^1_{,3} + \sin^2\theta T_2\xi^3_{,1} = 0, \tag{13}$$

$$T_{2,1}\xi^1 + 2T_2\xi^2_{,2} = 0, \tag{14}$$

$$\xi^2_{,3} + \sin^2\theta\xi^3_{,2} = 0, \tag{15}$$

$$T_{2,1}\xi^1 + 2T_2\cot\theta\xi^2 + 2T_2\xi^3_{,3} = 0, \tag{16}$$

where $T_3 = \sin^2\theta T_2$. It is to be noticed that we are using the notation $T_{aa} = T_a$. We solve these equations for the degenerate as well as the non-degenerate case. The nature of the solution of these equations changes when one (or more) T_a is zero. The nature changes even if $T_a \neq 0$ but $T_{a,1} = 0$.

3 Matter Collineations in the Degenerate Case

In this section we solve MC equations (7)-(16) when at least one of $T_a = 0$. First, we consider the trivial case, where $T_a = 0$. In this case, Eqs.(7)-(16) are identically satisfied and thus every vector field is an MC.

The other possibilities can be classified in three main cases:

(*1*) when only one of the $T_a \neq 0$,
(*2*) when exactly two of the $T_a \neq 0$,
(*3*) when exactly three of the $T_a \neq 0$.

Case (*1*): In this case, there could be only two possibilities:

($1a$) : $T_0 \neq 0, \quad T_i = 0 \quad (i = 1, 2, 3)$, ($1b$) : $T_1 \neq 0, \quad T_j = 0 \quad (j = 0, 2, 3)$.

The case ($1a$) is trivial and we get either (i) $T_0 = constant \neq 0$ or (ii) $T_0 \neq constant$. For the first possibility, we have

$$\xi = c_0 \partial_t + \xi^i(x^a)\partial_i, \tag{17}$$

where c_0 is a constant. For the second possibility, we obtain

$$\xi = f(t)\partial_t - \frac{\dot{f(t)}}{[\ln \sqrt{T_0}]'}\partial_r + \xi^\ell(x^a)\partial_\ell, \quad \ell = 2, 3. \tag{18}$$

In the case ($1b$), MC Eqs.(7), (9), (10) and (14)-(16) are identically satisfied while Eqs.(8), (12), (13) yield that $\xi^1 = \xi^1(r)$ only. Using Eq.(11), we have the following solution

$$\xi = \frac{c_0}{\sqrt{T_1}}\partial_r + \xi^j(x^a)\partial_j. \tag{19}$$

We have seen that in all subcases of the case (1), we obtain infinite number of MCs.

Case (*2*): This case implies the following two possibilities:

($2a$) : $T_k = 0, \quad T_\ell \neq 0$, ($2b$) : $T_k \neq 0, \quad T_\ell = 0$,

where $k = 0, 1$ and $\ell = 2, 3$ which are valid through this case.

The case ($2a$) explores further two possibilities i.e. (i) $T_2 = constant \neq 0$ and (ii) $T_2 \neq constant$. In the first option, we have the following solution of the MC equations

$$\xi^k = \xi^k(t, r, \theta, \phi), \quad \xi^2 = c_1 \cos\phi + c_2 \sin\phi, \quad \xi^3 = -\cot\theta(c_1 \sin\phi - c_2 \cos\phi) + c_0, \tag{20}$$

where c_0, c_1, c_2 are constants. Thus we can write

$$\xi = \xi^k(x^a)\partial_k + c_1(\cos\phi\partial_\theta - \cot\theta \sin\phi\partial_\phi) + c_2(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_0\partial_\phi. \tag{21}$$

We see that this contains the KVs associated with the usual spherical symmetry given in Appendix B.

In the case ($2aii$), MCs turn out to be

$$\xi = \xi^0(x^a)\partial_t + c_1(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_2(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_0\partial_\phi. \tag{22}$$

We again have the usual three KVs in addition to arbitrary MCs in the t direction.

For the case ($2b$), Eqs.(14)-(16) are identically satisfied and from the remaining Eqs.(7)-(13), we have

$$\frac{\ddot{A}}{A} = \frac{T_0}{T_1}\left(\frac{T_{0,1}}{2T_0\sqrt{T_1}}\right)' = \alpha, \tag{23}$$

where α is a separation constant. From Eq.(22), three possibilities arise

(i) $\alpha > 0$, (ii) $\alpha = 0$, (iii) $\alpha < 0$.

Solving MC equations for the case (*2bi*), we have the following set of MCs

$$\xi^0 = -\frac{T_{0,1}}{2T_0\sqrt{T_1}\sqrt{\alpha}}(c_1 \sin h\sqrt{\alpha}t + c_2 \cos h\sqrt{\alpha}t) + c_0, \tag{24}$$

$$\xi^1 = \frac{c_1 \cos h\sqrt{\alpha}t + c_2 \sin h\sqrt{\alpha}t}{\sqrt{T_1}}, \quad \xi^\ell = \xi^\ell(t, r, \theta, \phi). \tag{25}$$

In the case of (*2bii*), we obtain the following solution

$$\xi^0 = -\beta(c_1\frac{t^2}{2} + c_2 t) - c_1 \int \frac{\sqrt{T_1}}{T_0} dr + c_0, \tag{26}$$

$$\xi^1 = \frac{c_1 t + c_2}{\sqrt{T_1}}, \quad \xi^\ell = \xi^\ell(t, r, \theta, \phi), \tag{27}$$

where β is an integration constant which can be zero or non-zero and is given by

$$\frac{T_{0,1}}{2T_0\sqrt{T_1}} = \beta, \tag{28}$$

For the case of (*2biii*), solution of MC equations becomes

$$\xi^0 = \sqrt{p}(c_1 \sin \sqrt{p}t - c_2 \cos \sqrt{p}t) \int \frac{\sqrt{T_1}}{T_0} dr - c_0, \tag{29}$$

$$\xi^1 = \frac{c_1 \cos \sqrt{p}t + c_2 \sin \sqrt{p}t}{\sqrt{T_1}}, \quad \xi^\ell = \xi^\ell(t, r, \theta, \phi), \tag{30}$$

where $\alpha = -p$ and $p > 0$. When T_0 and T_1 do not satisfy Eq.(22), we have the following solution

$$\xi = c_0 \partial_t + \xi^\ell(x^a)\partial_\ell. \tag{31}$$

Again we see that all the possibilities of the case (2) give infinite-dimensional MCs.

Case (*3*): In this case, we have the following two possibilities:

$(3a) : T_0 = 0, \quad T_i \neq 0, \quad (3b) : T_1 = 0, \quad T_j \neq 0.$

For the case (*3a*), Eq.(7) is identically satisfied and Eqs.(8)-(10) respectively give $\xi^i = \xi^i(r, \theta, \phi)$, $(i = 1, 2, 3)$. From the remaining equations, we have the following constraint

$$\frac{A_{,22}}{A} = \frac{T_2}{\sqrt{T_1}} \left(\frac{T_{2,1}}{2T_2\sqrt{T_1}} \right)' = \alpha, \tag{32}$$

where α is a separation constant. This implies that we have three different possibilities:

$(i)\ \ \alpha > 0, \quad (ii)\ \ \alpha = 0, \quad (iii)\ \ \alpha < 0.$

The case (*3ai*) gives the same MCs as for the case (2aii).

For the case (*3aii*), we have two possibilities depending upon the value of constraint $\beta=\frac{T_{2,1}}{2T_2\sqrt{T_1}}$. When $\beta=0$, we have the following MCs

$$\xi=\xi^0(x^a)\partial_t+c_0\frac{1}{\sqrt{T_1(r)}}\partial_r+c_1(\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi)+c_2(\sin\phi\partial_\theta+\cot\theta\cos\phi\partial_\phi)+c_3\partial_\phi. \quad (33)$$

This again yields infinite-dimensional MCs in addition to the usual spherical symmetry KVs. For $\beta\neq 0$, MCs turn out to be of the case (2aii).
The case (*3aiii*) gives the following MCs

$$\begin{aligned}\xi=\xi^0\partial_t+c_0(\frac{\cos\theta}{\sqrt{T_1(r)}}\partial_r-\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}\sin\theta\partial_\theta)\\
+c_1[\frac{\sin\theta\cos\phi}{\sqrt{T_1(r)}}\partial_r+\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}(\cos\theta\cos\phi\partial_\theta+\sin\phi\partial_\phi)]\\
+c_2[\frac{\cos\theta\sin\phi}{\sqrt{T_1(r)}}\partial_r+\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}(\cos\theta\sin\phi\partial_\theta-\cos\phi\partial_\phi)]\\
+c_3\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}(\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi)\\
+c_4\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}(\sin\phi\partial_\theta+\cot\theta\cos\phi\partial_\phi)\\
+c_5\frac{T_2'(r)}{2T_2(r)\sqrt{T_2(r)}}\partial_\phi.\end{aligned} \quad (34)$$

This again provides infinite-dimensional MCs in addition to the usual KVs.
The case (*3b*) deals with the constraints $T_0\neq 0,\ \ T_1=0$ and $T_2\neq 0$. In addition to these, we can have the following constraints

(*i*) $T_{j,1}\neq 0$, (*ii*) $T_{0,1}=0,\ T_{2,1}\neq 0$,

(*iii*) $T_{0,1}\neq 0,\ T_{2,1}=0$, (*iv*) $T_{0,1}=0,\ \ T_{2,1}=0$.

In case (*3bi*), from MC Eqs.(7)-(16), we obtain the following MCs

$$\xi=c_0\partial_t+c_1(\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi)+c_2(\sin\phi\partial_\theta+\cot\theta\cos\phi\partial_\phi)+c_3\partial_\phi \quad (35)$$

which gives four independent MCs. This case is worth mentioning as we have found finite number of MCs even for the degenerate energy-momentum tensor. In case (*3bii*), Eq.(11) becomes identity while Eqs.(7) and (8) give $\xi^0=\xi^0(\theta,\phi)$. Also, Eqs.(12) and (13) respectively show that ξ^2 and ξ^3 are functions of t,θ,ϕ. From the remaining MC equations, we have the following solution

$$\begin{aligned}\xi=c_0\partial_t+c_1(\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi)+c_2(\sin\phi\partial_\theta+\cot\theta\cos\phi\partial_\phi)+c_3\partial_\phi\\
-\left[\frac{1}{\ln\sqrt{T_2}}\frac{\partial}{\partial\theta}\{[f_+(z)+f_-(\bar{z})]\sin\theta\}\right]\partial_r+[f_+(z)+f_-(\bar{z})]\sin\theta\partial_\theta\\
-[f_+(z)+f_-(\bar{z})]\partial_\phi,\end{aligned} \quad (36)$$

where $z = \ln \mid \csc\theta - \cot\theta \mid + \iota\phi$ and $f_+(z)$ and $f_-(\bar{z})$ are arbitrary functions of z and $\bar{z}$ (the complex conjugate of z) such that their sum is real and difference is imaginary.
For the case (*3biii*), one obtains the following MCs

$$\xi = f(t)\partial_t + \frac{-\dot{f}(t)}{[\ln\sqrt{T_0}]'}\partial_r + c_1(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_2(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_3\partial_\phi \qquad (37)$$

which gives an infinite number of MCs.
The case (*3biv*) yields the following MC vectors

$$\xi = c_0\partial_t + f(x^a)\partial_r + c_1(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_2(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_3\partial_\phi. \qquad (38)$$

Thus one has an infinite number of independent MCs in all subcases of of the case (3) except for (3bi) which has four independent MCs.

4 Matter Collineations in the Non-degenerate Case

In this section we shall solve MC equations when the energy-momentum tensor is non-degenerate, i.e, $T_a \neq 0$ since $det(T_a) \neq 0$.
If we solve MC Eqs.(7)-(16), after some tedious algebra, we arrive at the following solution

$$\xi^0 = -\frac{T_2\sin\theta}{T_0}[\dot{A}_1\sin\phi - \dot{A}_2\cos\phi] + \frac{T_2}{T_1}\dot{A}_3\cos\theta + A_4, \qquad (39)$$

$$\xi^1 = -\frac{T_2\sin\theta}{T_1}[A_1'\sin\phi - A_2'\cos\phi] + \frac{T_2}{T_1}A_3'\cos\theta + A_5, \qquad (40)$$

$$\xi^2 = \cos\theta[A_1\sin\phi - A_2\cos\phi] + A_3\sin\theta + c_1\sin\phi - c_2\cos\phi, \qquad (41)$$

$$\xi^3 = \mathrm{cosec}\theta[A_1\cos\phi + A_2\sin\phi] + [c_1\cos\phi + c_2\sin\phi]\cot\theta + c_0, \qquad (42)$$

where c_0, c_1 and c_2 are arbitrary constants and $A_\mu = A_\mu(t,r), \mu = 1,2,3,4,5$. These ξ^a are satisfied subject to the following differential constraints on A_μ

$$2T_1\ddot{A}_i + T_{0,1}A_i' = 0, \quad i = 1,2,3, \qquad (43)$$

$$2T_0\dot{A}_4 + T_{0,1}A_5 = 0, \qquad (44)$$

$$2T_2\dot{A}_i' + T_0\left(\frac{T_2}{T_0}\right)'\dot{A}_i = 0, \qquad (45)$$

$$T_0A_4' + T_{11}\dot{A}_5 = 0, \qquad (46)$$

$$\left\{T_{1,1}\frac{T_2}{T_1} + 2T_1\left(\frac{T_2}{T_1}\right)'\right\}A_i' + 2T_2A_i'' = 0, \qquad (47)$$

$$T_{1,1}A_5 + 2T_1A_5' = 0, \qquad (48)$$

$$T_{2,1}A_i' + 2T_1A_i = 0, \quad c_0 = 0, \tag{49}$$

$$T_{2,1}A_5 = 0. \tag{50}$$

Now the problem of working out MCs for all possibilities of A_i, A_4, A_5 is reduced to solving the set of Eqs.(39)-(42) subject to the above constraints. We start the classification of MCs by considering the constraint Eq.(50). This can be satisfied for three different possible cases.

$$\begin{array}{lll} (\mathit{1}) & T_{2,1} = 0, & A_5 \neq 0, \\ (\mathit{2}) & T_{2,1} \neq 0, & A_5 = 0, \\ (\mathit{3}) & T_{2,1} = 0, & A_5 = 0. \end{array}$$

Case *(1)*: In this case, all the constraints remain unchanged except (43), (47) and (49). Thus we have

$$\dot{A}_i' - \frac{1}{2}\frac{T_{0,1}}{T_0}\dot{A}_i = 0, \tag{51}$$

$$A_i'' - \frac{T_{1,1}}{2T_1}A_i' = 0, \tag{52}$$

$$T_1A_i = 0. \tag{53}$$

The last equation is satisfied only if $A_i = 0$. As a result, all the differential constraints involving A_i and its derivatives disappear identically and we are left with Eqs.(44), (46) and (48) only. Now integrating constraint Eq.(48) w.r.t. r and replacing the value of A_5 in constraint Eq.(44), we have

$$T_{0,1}\frac{A(t)}{\sqrt{T_1}} + 2T_0\dot{A}_4 = 0,$$

where $A(t)$ is an integration function. This can be satisfied for the following two possibilities:

$$(a) \;\; T_{0,1} = 0, \quad \dot{A}_4 = 0, \quad (b) \;\; T_{0,1} \neq 0, \quad \dot{A}_4 \neq 0.$$

For the case (1a), after some algebra, we arrive at the following set

$$\begin{aligned} \xi &= c_0\partial_t + c_4[\{\frac{-1}{a}\int\sqrt{T_1}dr\partial_t + \frac{t}{\sqrt{T_1}}\}\partial_r] + c_5\frac{1}{\sqrt{T_1}}\partial_r \\ &+c_1(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + c_2(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) + c_3\partial_\phi, \end{aligned} \tag{54}$$

where $a, c_0, c_1, c_2, c_3, c_4, c_5$ are arbitrary constants. This shows that we have six MCs.

In the case (1b), we have $\dot{A}_4 \neq 0$ and $T_{0,1} = 0$. Solving Eqs.(44) and (46) and rearranging terms, we get

$$\frac{\ddot{A}}{A} = \frac{1}{2}\left[\frac{T_{0,1}}{T_0\sqrt{T_1}}\right]'\frac{T_0}{\sqrt{T_1}} = \alpha, \tag{55}$$

where α is a separation constant and this gives the following three possible cases:

$$(i)\quad \alpha<0,\quad (ii)\quad \alpha=0,\quad (iii)\quad \alpha>0.$$

First we take $\alpha<0$ and assume that $\alpha=-p$, where p is positive, then Eq.(55) yields

$$A=A_{11}\cos\sqrt{p}t+A_{12}\sin\sqrt{p}t$$

and also

$$\frac{1}{2}\left[\frac{T_{0,1}}{T_0\sqrt{T_1}}\right]'=-\frac{p\sqrt{T_1}}{T_0}. \tag{56}$$

After some algebra, we can write the values of A_4 and A_5 given as

$$A_4=\sqrt{p}\int\frac{\sqrt{T_1}}{T_0}dr(A_{11}\sin\sqrt{p}t-A_{12}\cos\sqrt{p}t)+c_0,$$
$$A_5=\frac{A_{11}\cos\sqrt{p}t+A_{12}\sin\sqrt{p}t}{\sqrt{T_1}}.$$

Substituting these values in Eqs.(39)-(42) and using the constraints $A_{11}=0=A_{12}$, we obtain four independent MCs which are exactly similar to the case (3bi).
The subcase (1bii) gives

$$A(t)=c_3t+c_4$$

and

$$\frac{T_{0,1}}{T_0\sqrt{T_1}}=\beta, \tag{57}$$

where β is an integration constant which yields the following two possibilities

$$(*)\quad \beta\neq 0,\qquad (**)\quad \beta=0.$$

The first possibility implies that

$$T_0=\beta_0e^{\beta\int\sqrt{T_1}dr},$$

where β_0 is an integration constant. Now we solve Eqs.(48) and (50) by using this constraint, we can get the following set

$$\xi^0=-\frac{c_3}{\beta_0}\int\frac{\sqrt{T_1}}{e^{\beta\int\sqrt{T_1}dr}}dr+c_0,\quad \xi^1=\frac{c_3t+c_4}{\sqrt{T_1}},$$
$$\xi^2=c_1\sin\phi-c_2\cos\phi,\quad \xi^3=c_1\cos\phi+c_2\sin\phi. \tag{58}$$

This gives five independent MCs.
For the case (1bii**), $T_0=constant$. Using this fact Eq.(48) yields $A_4=g(r)$. Thus we have the solution

$$\xi^0=-\frac{c_3}{b}\int\sqrt{T_1}dr+c_0,\quad \xi^1=\frac{c_3t+c_4}{\sqrt{T_1}},$$
$$\xi^2=c_1\sin\phi-c_2\cos\phi,\quad \xi^3=c_1\cos\phi+c_2\sin\phi. \tag{59}$$

We again have five MCs.
The case (1biii) will give the similar results as in the case (1bi).
Case (*2*): In this case, Eqs.(44) and (46) show that A_4 is a pure constant. Integration of constraint Eq.(49) w.r.t. r gives

$$A_i = c_{i1}(t)e^{-2\int \frac{T_1}{T_{2,1}}dr}, \tag{60}$$

where $c_{i1}(t)$ is integration function. Substituting this in Eq.(43), we have after some simplifications

$$\frac{\ddot{c}_{i1}(t)}{c_{i1}} = \frac{T_{0,1}}{T_{2,1}} = \alpha, \tag{61}$$

where α is a separation constant. From here we have three possibilities

$$(a)\quad \alpha > 0, \quad (b)\quad \alpha = 0, \quad (c)\quad \alpha < 0.$$

In the case (2a), Eq.(61) gives

$$c_{i1} = a_{i1}\cosh\sqrt{\alpha}t + a_{i2}\sinh\sqrt{\alpha}t.$$

Substituting this value in Eq.(60), we get

$$A_i = (a_{i1}\cosh\sqrt{\alpha}t + a_{i2}\sinh\sqrt{\alpha}t)e^{-2\int \frac{T_1}{T_{2,1}}dr}.$$

Replacing this value in Eqs.(39)-(42) and then substituting the resulting values of ξ^a in MC Eq.(8), there are two possibilities either $a_{1i} = a_{2i} = a_{3i} = 0$ or

$$-4T_0T_1T_2 + T_0T_{2,1}^2 - T_0T_2T_{0,1}^2 = 0.$$

For the first possibility we obtain the result as for the case (1bi). For the second possibility subject to the constraint

$$T_2T_{1,1}T_{2,1} - 4T_1^2T_2T_{2,1} + 2T_1T_{2,1}^2 - 2T_1T_2T_{2,11} = 0,$$

we have the following results

$$\begin{aligned}
\xi^0 &= \frac{T_2}{T_0}e^{-2\int \frac{T_1}{T_{2,1}}dr}\sqrt{\alpha}[-\sin\theta\{(a_{11}\sinh\sqrt{\alpha}t + a_{12}\cosh\sqrt{\alpha}t)\sin\phi \\
&\quad -(a_{21}\sinh\sqrt{\alpha}t + a_{22}\cosh\sqrt{\alpha}t)\cos\phi\} + \cos\theta(a_{31}\sinh\sqrt{\alpha}t \\
&\quad +a_{32}\cosh\sqrt{\alpha}t)] + c_0, \\
\xi^1 &= \frac{-2T_2}{T_{2,1}}e^{-2\int \frac{T_1}{T_{2,1}}dr}[-\sin\theta\{(a_{11}\cosh\sqrt{\alpha}t + a_{12}\sinh\sqrt{\alpha}t)\sin\phi \\
&\quad -(a_{21}\cosh\sqrt{\alpha}t + a_{22}\sinh\sqrt{\alpha}t)\cos\phi\} + \cos\theta(a_{31}\cosh\sqrt{\alpha}t \\
&\quad +a_{32}\sinh\sqrt{\alpha}t)] + c_0, \\
\xi^2 &= (e^{-2\int \frac{T_1}{T_{2,1}}dr})[\cos\theta\{(a_{11}\cosh\sqrt{\alpha}t + a_{12}\sinh\sqrt{\alpha}t)\sin\phi \\
&\quad -(a_{21}\cosh\sqrt{\alpha}t + a_{22}\sinh\sqrt{\alpha}t)\cos\phi\} + \sin\theta(a_{31}\cosh\sqrt{\alpha}t \\
&\quad +a_{32}\sinh\sqrt{\alpha}t)] + (c_1\sin\phi - c_2\cos\phi), \\
\xi^3 &= \cot\theta[c_1\cos\phi + c_2\sin\phi] + c_3 + \csc\theta(e^{-2\int \frac{T_1}{T_{2,1}}dr})[(a_{11}\cosh\sqrt{\alpha}t \\
&\quad +a_{12}\sinh\sqrt{\alpha}t)\cos\phi + (a_{21}\cosh\sqrt{\alpha}t + a_{22}\sinh\sqrt{\alpha}t)\sin\phi].
\end{aligned} \tag{62}$$

For the case (2b), Eq.(61) gives $T_0 = constant$ and

$$c_{i1} = a_{i1}t + a_{i2},$$

Using the value of c_{i1} in Eq.(60), we obtain

$$A_i = (a_{i1}t + a_{i2})e^{-2\int \frac{T_1}{T_{2,1}}dr}. \tag{63}$$

Plugging these values in Eqs.(39)-(42) and re-labelling

$$a_{11} = c_4, \quad a_{21} = c_5, \quad a_{31} = c_6, \quad a_{12} = c_7, \quad a_{22} = c_8 \quad a_{32} = c_9,$$

we obtain the following MCs

$$\begin{aligned}
\xi^0 &= -\frac{T_2}{a}\left[\{c_4 \sin\phi - c_5 \cos\phi\}\sin\theta - c_6\cos\theta\right]e^{-2\int \frac{T_1}{T_{2,1}}dr} + c_0,\\
\xi^1 &= 2\frac{T_2}{T_{2,1}}[\{(c_4t + c_7)\sin\phi - (c_5t + c_8)\cos\phi\}\sin\theta - (c_6t\\
&\quad + c_9)\cos\theta]e^{-2\int \frac{T_1}{T_{2,1}}dr},\\
\xi^2 &= [\{(c_4t + c_7)\sin\phi - (c_5t + c_8)\cos\phi\}\cos\theta + (c_6t + c_9)\sin\theta]e^{-2\int \frac{T_1}{T_{2,1}}dr}\\
&\quad + (c_1\sin\phi - c_2\cos\phi),\\
\xi^3 &= \cot\theta[c_1\cos\phi + c_2\sin\phi] + c_3 + \csc\theta[(c_4t + c_7)\cos\phi + (c_5t + c_8)\sin\phi].
\end{aligned} \tag{64}$$

In the case (2c), Eq.(61) gives

$$c_{i1} = a_{i1}\cos\sqrt{\alpha}t + a_{i2}\sin\sqrt{\alpha}t.$$

Substituting this value in Eq.(60), we get

$$A_i = (a_{i1}\cos\sqrt{\alpha}t + a_{i2}\sin\sqrt{\alpha}t)e^{-2\int \frac{T_1}{T_{2,1}}dr}.$$

Further this case proceeds in the same way as the case (2a) and consequently the equivalent results.
Case (*3*): In this case, we have $T_2 = constant$ and $A_5 = 0$. This can be solved trivially and gives similar results as in the case (1bi).

5 Discussions and Conclusions

In the classification of spherically symmetric static spacetimes according to the nature of the energy-momentum tensor, we find that when the energy-momentum tensor is degenerate, Sec. (3), then there are many cases where the MCs are infinite-dimensional. It is very interesting to note that we have found a case (*3bi*) where the energy-momentum tensor is degenerate but the group of MCs is finite dimensional, i.e., there are four independent MCs. In the cases (1)-(3), we summarize some results in the following:

1. In this case, i.e., the rank of T_{ab} being 1, it is found that all the possibilities yield infinite-dimensional MCs.

2. In all subcases of this case, the rank of T_{ab} being 2. In subcase (2b), solving the equations $T_2 = 0$ and $T_3 = 0$, we obtain the Bertotti-Robinson I metric [13] given by

$$ds^2 = (B + r)^2 dt^2 - dr^2 - a^2 d\Omega^2, \quad (65)$$

where B and a are constants. This metric has six KVs but MCs are infinite dimensional.

3. For this case, the rank of T_{ab} is 3. The point worth mentioning in this case is the case (3bi) in which we have finite dimensionality of the group of MCs even if the energy-momentum tensor is degenerate. We obtain *four* MCs including the three KVs of spherical symmetric spacetimes.

Furthermore, we have dealt with the case when the energy-momentum tensor is non-degenerate (Sec.4). There are three main categories of the non-degenerate case which can be summarized as follows:

1. This case yields two possibilities. In the first possibility, we obtain six independent MCs in which four are the KVs and two are the proper MCs. The second possibility further has a division of three cases. The first case turns out similar to the case (3bi) of degenerate case which gives four MCs similar to the KVs. In the second case we have five independent MCs and the third case becomes the same as the first case.

2. In this case, we again have three possibilities. The first case has further two subcases in which first subcase reduces to (1bi) and the second subcase and the case (2b) gives 10 independent MCs which contain four KVs and dix proper MCs. The third case becomes the same as the case (1a).

3. This case trivially gives the same result as the case (1bi).

It is to be noticed that we have obtained MCs exactly similar to RCs [14] but with different constraint equations. If these constraint equations could be solved, then one would be able to find new exact solutions.

Appendix A

The surviving components of the Ricci tensor are

$$\begin{aligned}
R_0 &= \frac{1}{4}e^{v-\lambda}(2v'' + v'^2 - v'\lambda' + 2\mu' v'), \\
R_1 &= -\frac{1}{4}(2v'' + v'^2 - \lambda' v' + 4\mu'' + 2\mu'^2 - 2\mu'\lambda'), \\
R_2 &= -\frac{1}{4}e^{\mu-\lambda}(2\mu'' + 2\mu' - \mu'\lambda' + \mu' v') + 1, \\
R_3 &= R_2 \sin^2\theta, \quad (A1)
\end{aligned}$$

where prime '′' represents derivative w.r.t. r. The Ricci scalar is given by

$$R=\frac{1}{2}e^{-\lambda}(2v''+v'^2-v'\lambda'+2\mu'\lambda'+2\mu'v'+3\mu'^2+4\mu'')-2e^{-\mu}. \tag{A2}$$

Using Einstein field equations (1), the non-vanishing components of energy-momentum tensor T_{ab} are

$$\begin{aligned} T_0 &= \frac{e^{v-\lambda}}{4}(2\mu'\lambda'-3\mu'^2-4\mu'')+e^{v-\mu},\\ T_1 &= \frac{\mu'^2}{4}+\frac{\mu'v'}{2}-e^{\lambda-\mu},\\ T_2 &= \frac{e^{\mu-\lambda}}{4}(2\mu''+\mu'^2-\mu'\lambda'+\mu'v'+2v''+v'^2-v'\lambda'),\\ T_3 &= T_2\sin^2\theta. \end{aligned} \tag{A3}$$

Appendix B

Linearly independent KVs associated with the spherical symmetry of the spacetimes are

$$\begin{aligned} \xi_{(1)} &= \sin\phi\partial_\theta+\cot\theta\cos\phi\partial_\phi,\\ \xi_{(2)} &= \cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi,\\ \xi_{(3)} &= \partial_\phi,\ \xi_{(4)}=\partial_r \end{aligned} \tag{B1}$$

which are the generators of group G_4. The Lie algebra has the following commutators

$$[\xi_{(1)},\xi_{(2)}]=\xi_{(3)},\quad [\xi_{(2)},\xi_{(3)}]=\xi_{(1)},\quad [\xi_{(3)},\xi_{(1)}]=\xi_{(2)},\quad [\xi_{(4)},\xi_{(i)}]=0. \tag{B2}$$

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